Handbook of Fourier Analysis & Its Applications

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TABLE 2.3. Continuous time Fourier transform (CTFT) theorems. The Fourier transform, typically complex, can be expressed in rectangular (Cartesian) coordinates as $X(u) = R(u) + jI(u)$, or in polar coordinates, $X(u) = |X(u)|e^{j\angle X(u)}$. Convolution is defined by $x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$. The deterministic correlation integral is $x(t) \star h(t) = \int_{-\infty}^{\infty} x(\tau)h^*(\tau - t)d\tau = x(t) * h^*(-t)$. When $x = h$, this operation is deterministic autocorrelation.

Transform	$x(t)$	$\leftrightarrow X(u)$
scaling	$x(at)$	$\leftrightarrow \frac{1}{\|a\|} X\left(\frac{u}{a}\right)$
shift	$x(t-\tau)$	$\leftrightarrow X(u)e^{-j2\pi u\tau}$
modulation	$x(t)e^{j2\pi vt}$	$\leftrightarrow X(u-v)$
scale then shift	$x\left(\frac{t-\tau}{a}\right)$	$\leftrightarrow \frac{1}{\|a\|} X\left(\frac{u}{a}\right) e^{-j2\pi u\tau}$
shift then scale	$x\left(\frac{t}{a} - b\right)$	$\leftrightarrow \frac{1}{\|a\|} X\left(\frac{u}{a}\right) e^{-j2\pi uab}$
derivative	$\left(\frac{d}{dt}\right)^n x(t)$	$\leftrightarrow (j2\pi u)^n X(u)$
integral	$\int_{-\infty}^{t} x(\tau)d\tau$	$\leftrightarrow \frac{X(u)}{j2\pi u} + \frac{1}{2}X(0)\delta(u)$
conjugate	$x^*(t)$	$\leftrightarrow X^*(-u)$
transpose	$x(-t)$	$\leftrightarrow X(-u)$
inversion	$\int_{-\infty}^{\infty} X(u)e^{j2\pi ut} du$	$\leftrightarrow X(u)$
duality	$X(t)$	$\leftrightarrow x(-u)$
linearity	$ax_1(t) + bx_2(t)$	$\leftrightarrow aX_1(u) + bX_2(u)$
convolution	$x(t) * h(t)$	$\leftrightarrow X(u)H(u)$
correlation	$x(t) \star h(t)$	$\leftrightarrow X(u)H^*(u)$
real signals	if $x(t)$ is real	$\leftrightarrow X(u) = X^*(-u)$,
	$\Rightarrow R(u) = R(-u)$,	$I(u) = -I(-u)$
	$\|X(u)\| = \|X(-u)\|$,	$\angle\{X(u)\} = -\angle\{X(-u)\}$.
causal signals	$x(t) = x(t)\mu(t)$	$\leftrightarrow X(u) = \frac{-j}{\pi u} * X(u)$
	$\Rightarrow I(u) = \frac{-1}{\pi u} * R(u)$,	$R(u) = \frac{1}{\pi u} * I(u)$
Fourier series	$\sum_n c_n e^{j2\pi nt/T}$	$\leftrightarrow \sum_n c_n \delta\left(u - \frac{n}{T}\right)$
sampling theorem	$\sum_n x_n \mathrm{sinc}(2Bt - n)$	$\leftrightarrow \sum_n x_n e^{-j\pi u/B}\, \Pi\left(\frac{u}{2B}\right)$

TABLE 2.4. Some Fourier transform pairs. Additional Fourier transform pairs are in Tables 2.5, 4.1, 4.2 and 4.3. (Continued on the next page.)

$x(t)$	$\longleftrightarrow$	$X(u)$
$\delta(t)$	$\longleftrightarrow$	1
$\exp(j2\pi t)$	$\longleftrightarrow$	$\delta(u-1)$
$\cos(2\pi t)$	$\longleftrightarrow$	$\frac{1}{2}\left[\delta(u+1)+\delta(u-1)\right]$
$\sin(2\pi t)$	$\longleftrightarrow$	$\frac{j}{2}\left[\delta(u+1)-\delta(u-1)\right]$
$\Pi(t)$	$\longleftrightarrow$	$\mathrm{sinc}(t)$
$\exp(-t)\mu(t)$	$\longleftrightarrow$	$(1+j2\pi u)^{-1}$
$\cos(\pi t)\Pi(t)$	$\longleftrightarrow$	$\frac{1}{2}\left(\mathrm{sinc}\left(u+\frac{1}{2}\right)+\mathrm{sinc}\left(u-\frac{1}{2}\right)\right)$
$\mathrm{sinc}_k(t)$	$\longleftrightarrow$	$\Pi(u)/\mathrm{sinc}^k(u)$
$\mathrm{sinc}(t)\ \mathrm{sgn}(t)$	$\longleftrightarrow$	$-\frac{j}{\pi}\log\left\lvert\frac{u+\frac{1}{2}}{u-\frac{1}{2}}\right\rvert$
$d_p(t) := \left(\frac{d}{dt}\right)^p \mathrm{sinc}(t)$	$\longleftrightarrow$	$(j2\pi u)^p\ \Pi(u)$
$\mathrm{sinc}(t)\mu(t)$	$\longleftrightarrow$	$\frac{1}{2}\Pi(u)-\frac{j}{2\pi}\log\left\lvert\frac{u+\frac{1}{2}}{u-\frac{1}{2}}\right\rvert$
$\Lambda(t)$	$\longleftrightarrow$	$\mathrm{sinc}^2(u)$
$\Lambda(t)\ \mathrm{sgn}(t)$	$\longleftrightarrow$	$\frac{-j}{2\pi u}\ (1-\mathrm{sinc}(2u))$
$\mathrm{sgn}(t)$	$\longleftrightarrow$	$-j/(\pi u)$
$\frac{1}{t}$	$\longleftrightarrow$	$-j\pi\ \mathrm{sgn}(u)$
$\frac{1}{t}\Pi\left(\frac{t}{2}\right)$	$\longleftrightarrow$	$-j2\ \mathrm{Si}(2\pi u)$
$\lvert t\rvert^{-1/2}$	$\longleftrightarrow$	$-\lvert u\rvert^{-1/2}$
$\lvert t\rvert^{-1/2}\ \mathrm{sgn}(t)$	$\longleftrightarrow$	$-j\lvert u\rvert^{-1/2}\mathrm{sgn}(u)$
$t^{-1/2}\ \mu(t)$	$\longleftrightarrow$	$(-j2u)^{1/2}$
$\mu(t)$	$\longleftrightarrow$	$\frac{1}{2}\left(\delta(u)-\frac{j}{\pi u}\right)$
$\mathrm{e}^{-\lvert t\rvert}$	$\longleftrightarrow$	$2\left(1+(2\pi u)^2\right)^{-1}$
$\mathrm{e}^{-\lvert t\rvert}\mathrm{sgn}(t)$	$\longleftrightarrow$	$-j4\pi u\left(1+(2\pi u)^2\right)^{-1}$
$\mathrm{e}^{-\pi\lvert t\rvert}\mathrm{sinc}(t)$	$\longleftrightarrow$	$\frac{1}{\pi}\ \arctan\left(\frac{1}{2u^2}\right)$
$\lvert t\rvert\mathrm{e}^{-a\lvert t\rvert}$	$\longleftrightarrow$	$\sqrt{\frac{2}{\pi}}\ \frac{a^2-(2\pi u)^2}{(a^2+(2\pi u)^2)^2}$

TABLE 2.5.   Some Fourier transform pairs. (Continued from Table 2.4.)

$x(t)$	$\longleftrightarrow$	$X(u)$
$\mathrm{Si}(\pi t)$	$\longleftrightarrow$	$\dfrac{\Pi(u)}{j2u}$
$\cos(\pi t)/(\pi t)$	$\longleftrightarrow$	$j\mu\left(-u-\frac{1}{2}\right)-j\mu\left(u-\frac{1}{2}\right)$
$J_0(2\pi t)$	$\longleftrightarrow$	$\Pi\left(\frac{u}{2}\right)/\sqrt{1-u^2}$
$J_0(2\pi t)\mathrm{sgn}(t)$	$\longleftrightarrow$	$\dfrac{j}{\pi\sqrt{u^2-1}}\left(\Pi\left(\frac{u}{2}\right)-1\right)\mathrm{sgn}(u)$
$\mathrm{jinc}(t)$	$\longleftrightarrow$	$\sqrt{1-u^2}\;\Pi\left(\frac{u}{2}\right)$
$\dfrac{J_\nu(2\pi t)}{(2t)^\nu};\ \nu>-\dfrac{1}{2}$	$\longleftrightarrow$	$\dfrac{(\pi/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}(1-u^2)^{\nu-\frac{1}{2}}\,\Pi\left(\frac{u}{2}\right)$
$\mathrm{comb}(t)$	$\longleftrightarrow$	$\mathrm{comb}(u)$
$\sum_{n=0}^{\infty}\delta(t-n)$	$\longleftrightarrow$	$\frac{1}{2}\left(1-j\ \mathrm{cotan}(\pi u)\right)$
$\sum_{n=-N}^{N}\ \delta(t-n)$	$\longleftrightarrow$	$(2N+1)\ \mathrm{array}_{2N+1}(u)$
$\exp(-\pi t^2)$	$\longleftrightarrow$	$\exp(-\pi u^2)$
$\exp(\,jat^2)$	$\longleftrightarrow$	$\sqrt{j\pi/a}\ \exp(-j(\pi u)^2/a)$
$\mathrm{sech}(\pi t)$	$\longleftrightarrow$	$\mathrm{sech}(\pi u)$
$\mathrm{sech}(\pi t)\ \tanh(\pi t)$	$\longleftrightarrow$	$-j\pi u\ \mathrm{sech}(\pi u)$
$\mathrm{sech}^2(\pi t)$	$\longleftrightarrow$	$2u\ \mathrm{cosech}(\pi u)$
$\tanh(\pi t)$	$\longleftrightarrow$	$-j\ \mathrm{cosech}(\pi u)$
$t^{k-1}\mathrm{e}^{-at}\mu(t);\ \ a>0$	$\longleftrightarrow$	$\dfrac{\Gamma(k)}{(a+j2\pi u)^k}$
$J_n(2\pi t)$	$\longleftrightarrow$	$\dfrac{1}{\pi}\ \dfrac{j^{-n}}{\sqrt{1-u^2}}\ T_n(u)\Pi\left(\frac{u}{2}\right)$
$j_n(2\pi t)$	$\longleftrightarrow$	$\frac{1}{2}\,j^{-n}P_n(u)\Pi\left(u/2\right)$
$e^{-\pi t^2}H_n(\sqrt{2\pi}\ t)$	$\longleftrightarrow$	$j^{-n}e^{-\pi u^2}H_n(\sqrt{2\pi}\ u)$

TABLE 2.6. Discrete time Fourier transform (DTFT) theorems. Discrete time convolution is defined by $x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m]$. Correlation is $x[n] \star h[n] = \sum_{m=-\infty}^{\infty} x[m]\, h^*[n-m] = x[n] * h^*[-n]$. The notation "$\int_1$" means integration over any unit interval.

Transform periodicity	$x[n]$	$\leftrightarrow$	$X_D(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi nf}$ $= X_D(f-p);\, p = 0, \pm 1, \pm 2, \ldots$
shift	$x[n-k]$	$\leftrightarrow$	$X_D(f) e^{-j2\pi kf}$
modulation	$x[n] e^{j2\pi n\nu}$	$\leftrightarrow$	$X_D(f-\nu)$
cumulative sum	$\sum_0^n x[k];\ x[n]$ is causal	$\leftrightarrow$	$\frac{X_D(u)}{e^{j2\pi f}-1}$
conjugate	$x^*[n]$	$\leftrightarrow$	$X_D^*(-f)$
transpose	$x[-n]$	$\leftrightarrow$	$X_D(-f)$
inversion	$x[n] = \int_1 X(f) e^{j2\pi nf} df$	$\leftrightarrow$	$X_D(f)$
linearity	$ax[n] + by[n]$	$\leftrightarrow$	$aX_D(f) + bY_D(f)$
convolution	$x[n] * h[n]$	$\leftrightarrow$	$X_D(f) H_D(f)$
circular convolution	$x[n]h[n]$	$\leftrightarrow$	$\int_{-\infty}^{\infty} X_D(\nu)\Pi(\nu) H_D(f-\nu) d\nu$ $= \int_1 X_D(\nu) H_D(f-\nu) d\nu$
correlation	$x(t) \star h(t)$	$\leftrightarrow$	$X_D(u) H_D^*(u)$

R.J. Marks II, Handbook of Fourier Analysis and Its Applications, Oxford University Press, (2008).

TABLE 3.1. Continuous time convolution algebra.

Property	Continuous		
definition	$x(t) * h(t)$	$=$	$\int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$
identity element	$x(t) * \delta(t)$	$=$	$x(t)$
commutative	$x(t) * h(t)$	$=$	$h(t) * x(t)$
associative	$x(t) * \{y(t) * h(t)\}$	$=$	$\{x(t) * y(t)\} * h(t)$
distributive	$x(t) * \{y(t) + h(t)\}$	$=$	$\{x(t) * y(t)\}$
			$+\{x(t) * h(t)\}$
shift	$x(t-\tau) * h(t)$	$=$	$x(t) * h(t-\tau)$
frequency response	$h(t) * \mathrm{e}^{j2\pi ut}$	$=$	$H(u)\mathrm{e}^{j2\pi ut}$
Fourier transform	$x(t) * h(t)$	$\leftrightarrow$	$X(u)H(u)$
derivative	$\frac{d}{dt}\left[x(t) * h(t)\right]$	$=$	$\frac{dx(t)}{dt} * h(t)$
		$=$	$x(t) * \frac{dh(t)}{dt}$

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TABLE 3.2. Discrete time convolution algebra.

Property	Discrete		
definition	$x[n] * h[n]$	$=$	$\sum_{-\infty}^{\infty} x[m]h[n-m]$
identity element	$x[n] * \delta[n]$	$=$	$x[n]$
commutative	$x[n] * h[n]$	$=$	$h[n] * x[n]$
associative	$x[n] * \{y[n] * h[n]\}$	$=$	$\{x[n] * y[n]\} * h[n]$
distributive	$x[n] * \{y[n] + h[n]\}$	$=$	$\{x[n] * y[n]\}$
			$+\{x[n] * h[n]\}$
shift	$x[n-m] * h[n]$	$=$	$x[n] * h[n-m]$
frequency response	$h[n] * \mathrm{e}^{j2\pi nf}$	$=$	$H_D(f)\mathrm{e}^{j2\pi nf}$
Fourier transform	$x[n] * h[n]$	$\leftrightarrow$	$X_D(f)H_D(f)$

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TABLE 4.1. The probability density functions, characteristic functions, means and variance of some common continuous random variables are listed here. The characteristic function, $\Phi_X(u)$, is the Fourier transform of $f_X(x)$. (Continued in Table 4.2.)

Name	Parameters	$f_X(x)$	$\Phi_X(u)$	$\overline{X}$	σ_X^2
Uniform	$\overline{X}, R > 0$	$\frac{1}{R}\Pi\left(\frac{x-\overline{X}}{R}\right)$	$\text{sinc}(Ru)e^{-j2\pi u\overline{X}}$	$\overline{X}$	$\frac{R}{12}$
Triangle	$\overline{X}, R > 0$	$\frac{1}{R}\Lambda\left(\frac{x-\overline{X}}{R}\right)$	$\text{sinc}^2(Ru)e^{-j2\pi u\overline{X}}$	$\overline{X}$	$\frac{R^2}{6}$
Gaussian	$\overline{X}$	$\frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(x-\overline{X})^2}{2\sigma_X^2}}$	$e^{-j2\pi u\overline{X}}$	$\overline{X}$	σ_X^2
	$\sigma_X > 0$		$\times e^{-2(\pi u\sigma_X)^2}$		
Exponential	$\lambda > 0$	$\lambda e^{-\lambda x}\mu(x)$	$\frac{\lambda}{\lambda+j2\pi u}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Laplace	$\overline{X}$	$\frac{\alpha}{2}e^{-\alpha\lvert x-\overline{X}\rvert}$	$\frac{\alpha^2}{(2\pi u)^2+\alpha^2}$	$\overline{X}$	$\frac{2}{\alpha^2}$
	$\alpha > 0$		$\times e^{-j2\pi u\overline{X}}$		
Sech	$\alpha > 0$	$\alpha\ \text{sech}(\pi\alpha x)$	$\text{sech}(\pi u/\alpha)$	0	$\frac{1}{4\alpha^2}$
Pearson III	$\alpha, \beta > 0$	$\frac{1}{\beta\Gamma(p)}\left(\frac{x-\alpha}{\beta}\right)^{p-1}$	$\left(\frac{\beta}{\beta+j2\pi u}\right)^p$	$\alpha + \frac{p}{\beta}$	$p\beta^2$
	$p \in \mathbb{N}$	$\times e^{-\frac{x-\alpha}{\beta}}\mu(x-\alpha)$	$\times e^{-j2\pi u\alpha}$		

TABLE 4.2. Continued from Table 4.1. Notes. (1) The Erlang random variable (See Figure 4.5) is a special case of the gamma random variable with $\alpha = m \in \mathbb{N}$. (2) The chi-square ($\chi^2$) random variable (See Figure 4.4) is also a special case of the gamma with $\alpha = k/2$, $k \in \mathbb{N}$ and $\lambda = 1/2$. (3) $\Upsilon_{a,b}(u) = {}_1F_1(b+1, b+c+2, -j2\pi u)$. See Exercise 4.9 for the derivation. (4) The Cauchy random variable has undefined mean and variance.

Name	Parameters	$f_X(x)$	$\Phi_X(u)$	$\overline{X}$	σ_X^2
Gamma	$\lambda > 0, \alpha > 0$	$\frac{\lambda(\lambda x)^{\alpha-1}e^{-\lambda x}\mu(x)}{\Gamma(\alpha)}$	$\frac{1}{(1+j2\pi u/\lambda)^\alpha}$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$
Erlang[1]	$\lambda > 0, m \in \mathbb{N}$	$\frac{\lambda(\lambda x)^{m-1}e^{-\lambda x}\mu(t)}{(m-1)!}$	$\left(\frac{\lambda}{\lambda+j2\pi u}\right)^m$	$\frac{m}{\lambda}$	$\frac{m}{\lambda^2}$
Chi-Square[2]	$k \in \mathbb{N}$	$\frac{x^{-1+k/2}e^{-x/2}\mu(t)}{2^{k/2}\Gamma(k/2)}$	$\left(\frac{1}{1+j4\pi u}\right)^{k/2}$	k	$2k$
Beta[3]	$a, b > 0$	$\frac{x^b(1-x)^c}{B(b+1,c+1)}$	$(b+c+1)$	$\frac{b+1}{b+c+2}$	$\frac{b+1}{(b+c+2)^2}$
		$\times\ \Pi\left(x-\frac{1}{2}\right)$	$\times\ \Upsilon_{a,b}(u)$		$\times\frac{b+2}{b+c+3}$
Cauchy[4]	$\alpha > 0$	$\frac{\alpha/\pi}{x^2+\alpha^2}$	$e^{-2\pi\alpha\lvert u\rvert}$	no	no

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TABLE 4.3. Probability mass functions for common discrete random variables. The probability density function for discrete random variables of the lattice type are in (4.7). The characteristic function is in (4.8). The characteristic function, $\Phi_X(u)$, is the discrete time Fourier transform of p_k. Notes: (1) $0 \le p \le 1$ and $q := 1 - p$, (2) $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ (3) See Exercise 4.12. The negative binomial is stated more generally in some sources with probability mass $p_k = \Gamma(\rho + k)p^{\rho}(1-p)^k/(k!\ \Gamma(\rho))$ for $\rho > 0$. This is also referred to as the *Pólya distribution*. The entry in this table is then relegated to the status of a special case for $\rho = r$ = a positive integer and is called the *Pascal distribution*.

Name	Parameter(s)	p_k	$\Phi_X(u)$	$\overline{X}$	σ_X^2
Deterministic	a	$\delta[x-a]$	$e^{-j2\pi au}$	a	0
Bernoulli[1]	p	$p\ \delta[k] + q\ \delta[k-1]$	$p + qe^{-j2\pi u}$	p	pq
Uniform discrete	$N \in \mathbb{N}$	$\frac{1}{N};\ 0 \le n < N$	$e^{-j\pi(N-1)u}$ $\times \text{array}_N(u)$	$\frac{N-1}{2}$	$\frac{(N-1)(N+1)}{12}$
Binomial[1,2]	$n > 0, p$	$\binom{n}{k}p^k q^{n-k};$ $0 \le k \le n$	$(q + pe^{-j2\pi u})^n$	np	npq
Geometric[1]	p	$pq^k \mu[k]$	$\frac{p}{1-qe^{-j2\pi u}}$	$\frac{1}{p}$	$\frac{q}{p^2}$
Poisson	$\alpha > 0$	$\frac{\alpha^k e^{-\alpha}\mu[k]}{k!}$	$e^{\alpha(e^{-j2\pi u}-1)}$	α	α
Negative binomial[1,2,3]	p, r	$\binom{k-1}{r-1}p^r q^{k-r}\mu[k-r]$	$\left(\frac{p}{e^{j2\pi u}-q}\right)^r$	$\frac{r}{p}$	$\frac{rq}{p^2}$

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